The invariant measure for $SU(N)$

The invariant measure for integration over SU(N) takes the form of a direct product of a uniform intigration over the sphere S_{2N-1} and the invariant measure over $SU(N-1)$

Parameterization of $U \in SU(N)$

- the rows of the matrix U are three orthogonal complex vectors
- factor U into two pieces

$$U = \begin{pmatrix} 1 & \cdots \\ \vdots & g_{N-1} \end{pmatrix} g_s(\vec{z})$$

- $g_{N-1} \in SU(N-1)$
- $g_s \in SU(N)$ is a standard form with given top row $\vec{z}$ $\quad g_x(\vec{z}) = \begin{pmatrix} \vec{z} \\ \vdots \end{pmatrix}$
- $\vec{z}$ is a complex unit N-vector, $|\vec{z}|^2 = 1$

$\vec{z}$ defines a sphere S_{2N-1}

- parameterize

$$\vec{z} = \left(c_1 p_1, s_1 c_2 p_2, s_1 s_2 c_3 p_3, \ldots, \prod_i^{N-2} s_i \; c_{N-1} p_{N-1}, \prod_i^{N-1} s_i \; p_N \right)$$

- $c_i = \cos(\theta_i)$, $s_i = \sin(\theta_i)$, $p_i = e^{i\phi_i}$
- $2N - 1$ parameters: $\theta_{1,\ldots N-1}$, $\phi_{1,\ldots N}$
- $0 \le \theta_i < \pi/2 \qquad 0 \le \phi_i < 2\pi$

Uniform measure on the sphere

$$dS_{2N-1} = \frac{(2N-2)(2N-4)\ldots 2}{(2\pi)^N} (d\theta)(d\phi) \prod_{i=1}^{N-1} c_i s_i^{2N-2i-1}$$

Parameterize the standard form $g_s(\vec{z})$

$$g_s = \begin{pmatrix} 1 & 0 & \ldots & \\ 0 & 1 & \ldots & \\ \vdots & \vdots & \ddots & \\ & & & \prod_i p_i^* \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 & \ldots \\ -s_1 & c_1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & c_2 & s_2 & \ldots \\ 0 & -s_2 & c_2 & \ldots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix}$$

$$\ldots \begin{pmatrix} 1 & \vdots & \vdots & \vdots \\ \ldots & 1 & 0 & 0 \\ \ldots & 0 & c_{N-1} & s_{N-1} \\ \ldots & 0 & -s_{N-1} & c_{N-1} \end{pmatrix} \begin{pmatrix} p_1 & 0 & \ldots & \\ 0 & p_2 & \ldots & \\ \vdots & \vdots & \ddots & \\ & & & p_N \end{pmatrix}$$

This parameterization covers the group

- g_{N-1}: $(N-1)^2 - 1$ parameters
- S_{2N-1}: $2N-1$ parameters
- total: N^2-1 parameters of $SU(N)$

Measure for $SU(N)$

- $dg_N = f(\vec{z}, g_{N-1}) dS_{2N-1}\, dg_{N-1}$
- weight factor $f(\vec{z}, g_{N-1})$ to be determined

Left invariance over the $SU(N-1)$

- $f(\vec{z}, g_{N-1})$ cannot depend on g_{N-1}

Right invariance on $SU(2)$ subgroups

- can implement arbitrary rotations on $\vec{z}$
- measure does not depend on $\vec{z}$

$$dg_N = dS_{2N-1}\, dg_{N-1}$$

$\boxed{\text{Measure uniform over both } S_{2N-1} \text{ and } g_{N-1}}$

Example: $SU(3)$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_1^* p_2^* p_3^* p_4^* p_5^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_4 & 0 \\ 0 & 0 & p_5 \end{pmatrix}$$

$$\begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$$

Five phases and three angles

$$dg = \frac{3}{8\pi^5} c_1 c_2 c_3 s_1^2 s_2 s_3 \ (d\theta)(d\phi)$$

$$0 \le \theta_i < \pi \qquad 0 \le \phi_i < 2\pi$$

Periodicity and $\Pi_{2N-1}(SU(N))$

Map S_{2N-1} into the group in a smooth but non-contractable way

Try mapping the S_{2N-1} into the top row

- parameterization of g_s singular at the poles when $s_i = 0$
- resolve recursively by defining a new $\tilde{g}_s =$

$$\begin{pmatrix} p_1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & p_1^* \end{pmatrix} \begin{pmatrix} 1 & \ldots \\ \vdots & g_{N-1}^\dagger \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 & \ldots \\ -s_1 & c_1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & 1 \end{pmatrix} \begin{pmatrix} 1 & \ldots \\ \vdots & g_{N-1} \end{pmatrix}$$

- g_{N-1} submatrix is in $SU(N-1)$ and contains the remaining angles

At the north pole with $\theta_1 = 0$

- the second factor cancels the last
- things are smooth

But, still singular at the south pole

- cut out that pole
- keep going in θ_1 beyond $\pi/2$
- at $\theta_1 = \pi$ things are still singular
- finally at $\theta_1 = 2\pi$ get a non singular point

For a nontrivial mapping of S_{2N-1} into $SU(N)$

- take $\theta_1 = \tilde{\theta}_1/4$
- $\tilde{\theta}_1$ parameterizes the S_{2N-1} top row
- for $\Pi_{2N-1}(SU(N))$ we cover the range of $\tilde{\theta}_1$ 4 times

Recursing to the lower groups

- additional factors of 4 until $SU(2)$
- a $SU(2)$ matrix is entirely determined by its top row

To map an S_{2N-1} nontrivially into into $SU(N)$

- sweep over possible first rows 4^{N-2} times

Bott periodicity factor $= \ 4^{N-2}$